ELECTIVE 2
OPTIMAL CONTROL SYSTEMS
(ACE 326)

Background
Ref.1&2: Appendix A

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VECTOR & MATRIX ALGEBRA
OUTLINES

- **Matrices Definition**
- **Matrices Types**
- **Matrices Operations**
- **Solving n Linear Equations /n Unknowns**

VECTOR & MATRIX ALGEBRA
HISTORY

- **Linear algebra** is the branch of mathematics concerning vector spaces/linear mapping between such spaces

- Leibniz (1693) ⟶ Determinants

- Gabriel Cramer (1750-Cramer's rule) ⟶ solving linear systems

- Gauss 1777–1855 ⟶ solving linear systems by Gaussian elimination method

Gottfried Wilhelm Leibniz (1646-1716): German mathematician and philosopher
Gabriel Cramer (1704-1752): Swiss mathematician
Johann Carl Friedrich Gauss (1777-1855): German mathematician

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MATRICES DEFINITION

- A matrix is a rectangular array of quantities (real numbers/complex numbers/functions of variables)

row

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

column

Matrix $\boldsymbol{A} = \left[a_{ij}\right]_{(m \times n)}$

Elements of **A**

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MATRICES TYPES

- Rectangle matrix

$$\boldsymbol{A} = \left[a_{ij}\right]_{(m\times n)}$$

- Square matrix

$$m = n \longrightarrow \boldsymbol{A} = \left[a_{ij}\right]_{(n\times n)}$$

- Vector

$$m = 1 \longrightarrow \boldsymbol{A} = \left[a_{ij}\right]_{(1\times n)} = \mathbf{a} \text{ (row vector)}$$

$$n = 1 \longrightarrow \boldsymbol{A} = \left[a_{ij}\right]_{(m\times 1)} = \mathbf{a} \text{ (column vector)}$$

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MATRICES TYPES

- Upper triangular matrix

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

- Lower triangular matrix

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

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MATRICES TYPES---CONT.

- Null Matrix

$$\boldsymbol{A} = \left[a_{ij}\right]_{(m\times n)} = \mathbf{0}$$

$$\boldsymbol{AB} = \boldsymbol{BA} = \mathbf{0}$$

- Unity Matrix (Identity Matrix)

$$\boldsymbol{A} = \boldsymbol{I} = \boldsymbol{I}_n = [a_{ii}]_{(n\times n)} = 1; \;\; i = 1:n$$

- Singular Matrix/Nonsingular Matrix

$$|\boldsymbol{A}| = 0 \qquad\qquad |\boldsymbol{A}| \neq 0$$

Determinant of A-
Full rank

- Symmetric Matrix

$$\boldsymbol{A} = \boldsymbol{A}^T$$

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MATRICES OPERATIONS

- **Addition**

$$\boldsymbol{C}_{(m\times n)} = \boldsymbol{A} + \boldsymbol{B};\ c_{ij} = a_{ij} + b_{ij}$$

- Properties

1. Commutative ⟶ $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$

2. Associative ⟶ $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \mathbf{C}$

3. $A + C = B + C$ ⟶ $\boldsymbol{A} = \boldsymbol{B}$

4. $a_{ij} = b_{ij}$ ⟶ $\boldsymbol{A}_{(m\times n)} = \boldsymbol{B}_{(m\times n)}$

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MATRICES OPERATIONS

- **Multiplication**

$$\boldsymbol{C}_{(m\times p)} = \boldsymbol{A}_{(m\times n)}\boldsymbol{B}_{(n\times p)};\ c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

- Properties

1. Commutative ⟶ $\boldsymbol{AB} \neq \boldsymbol{BA}$
2. Associative ⟶ $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\mathbf{C}$
3. Distributive ⟶ $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\mathbf{AC}$

$$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\mathbf{BC}$$

$$(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{AC}+\boldsymbol{AD}+\boldsymbol{BC}+\boldsymbol{BD}$$

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MATRICES OPERATIONS

- **Transpose** of a matrix flips a matrix over its diagonal

$$A^T = A'$$

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

- Properties

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(\alpha A)^T = \alpha A^T; \alpha\ scalar$
4. $(AB)^T = B^T A^T$

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MATRICES OPERATIONS---CONT.

5. $\boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{a}\,.\,\boldsymbol{b} = \sum_{j=1}^{n} a_j\, b_j$

vectors

Dot product

$$(\boldsymbol{a}^T\boldsymbol{a})^{1/2} = \|\boldsymbol{a}\| = \left(\sum_{j=1}^{n} {a_j}^2\right)^{1/2}$$

Norm
vector length-scalar

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MATRICES OPERATIONS---CONT.

- **Determinant** is a scaling factor of a matrix

$$\det(\boldsymbol{A}) = |\boldsymbol{A}|$$

$$|\boldsymbol{A}| = \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}} a_{ij}\,\mathrm{cofac}(a_{ij}) \qquad \mathrm{cofac}(a_{ij}) = (-1)^{\boldsymbol{i}+\boldsymbol{j}}|\boldsymbol{A}_{\boldsymbol{ij}}|$$

for any $i \; or \; j$

- Properties
  1. $|\boldsymbol{A}^T| = |\boldsymbol{A}|$
  2. If a square matrix **A** has two identical columns (rows)/elements of a column (row) are zero $\longrightarrow$ $|\boldsymbol{A}| = 0$
  3. $|\boldsymbol{A}| = \prod_{i=1}^{n} a_{ii}$ $\longrightarrow$ upper/lower triangular matrix

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MATRICES OPERATIONS---CONT.

- **Inverse** of a matrix

$$(\boldsymbol{A})^{-\mathbf{1}} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\, adj(\boldsymbol{A})$$

$$adj(\boldsymbol{A}) = (\text{cofac}(a_{ij}))^T$$

- Properties

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1}$
3. $|\boldsymbol{A}|^{-1} = |\boldsymbol{A}^{-1}|$
4. $(\alpha\boldsymbol{A})^{-1} = \alpha^{-1}\boldsymbol{A}^{-1};\ \alpha\ \ nonzero\ scalar$

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MATRICES OPERATIONS---CONT.

- **Elementary row/column operations**

1. Interchange any two rows (columns).
2. Multiply any row (column) by a nonzero scalar.
3. Add to any row (column) a scalar multiple of another row (column).

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SOLVING n LINEAR EQUATIONS/n UNKNOWNS

- Linear equations are encountered in numerous engineering and scientific applications

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 6 \\
-x_1 + 6x_2 - 2x_3 &= 3
\end{aligned}
$$

$$
\begin{array}{ccc|c}
x_1 & x_2 & x_3 & \mathbf{b} \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 2 & 3 & 6 \\
-1 & 6 & -2 & 3
\end{array}\right]
$$

Dummy variables

Augmented matrix

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SOLVING n LINEAR EQUATIONS/n UNKNOWNS

- Let two linear equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_1$$

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Elimination Process →

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2$$

$$(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - a_{21}b_1$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

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SOLVING *n* LINEAR EQUATIONS/*n* UNKNOWNS

$$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|}, \qquad x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|}$$

$$\mathbf{B}_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

Cramer's rule

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SUMMARY

- **Solution of $n \times n$ system of linear equations**

  - $|A| \neq 0$ $\longrightarrow$ Unique solution
  - $|A| = 0$ $\longrightarrow$ Infinite number of solutions/No solution

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SOLVING m LINEAR EQUATIONS/ n UNKNOWNS

Matrix rank is largest nonsingular square submatrix of given matrix $\longrightarrow$ $\boldsymbol{r}$

- $n > m,\ r \leq m$ $\longrightarrow$ full row rank matrix- ∞ solutions
- $m > n,\ r \leq n$ $\longrightarrow$ full column rank matrix- no solution/ ∞ solutions
- $m = n,\ r = n$ $\longrightarrow$ nonsingular matrix- unique solution

Thank You for Your Attention