Elective 2
Optimal Control Systems
(ACE 326)

Lecture 4- Calculus-Based Optimization and Basic Concepts
Ref. 1: Chapter 4

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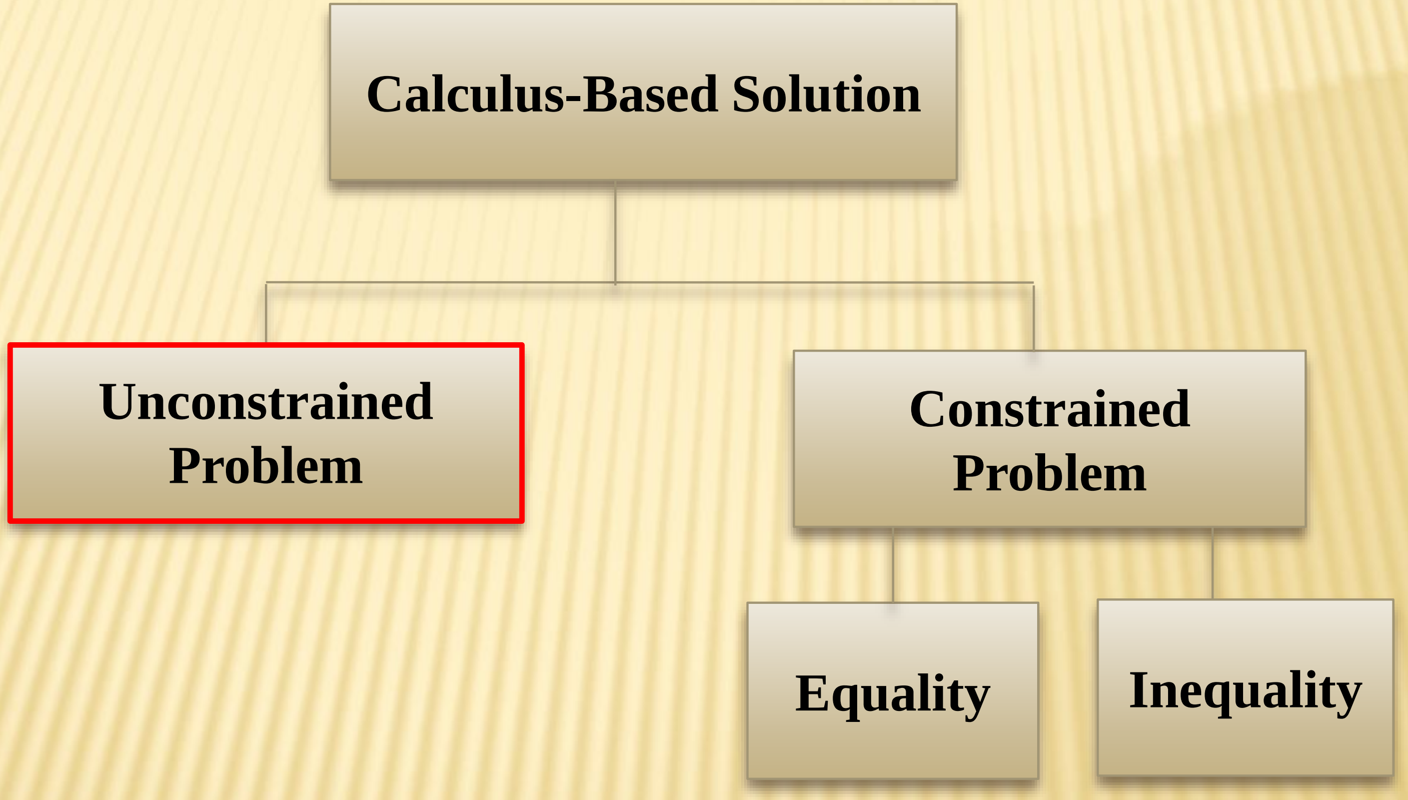
OUTLINES

- Local and global minima (maxima) for optimization problems
- Optimality conditions for unconstrained problems
  - Single variable optimization problem
  - Multi-variables optimization problem

OPTIMIZATION PROBLEM SOLUTION



GLOBAL & LOCAL MINIMA/MAXIMA

- **Global minimum: a function $f(x)$ of n variables has a global minimum at x^* if $f(x^*)$ is less than or equal to $f(x)$ at any x in the feasible set S.**

$$f(x^*) \leq f(x) \; \forall \, x \in S$$

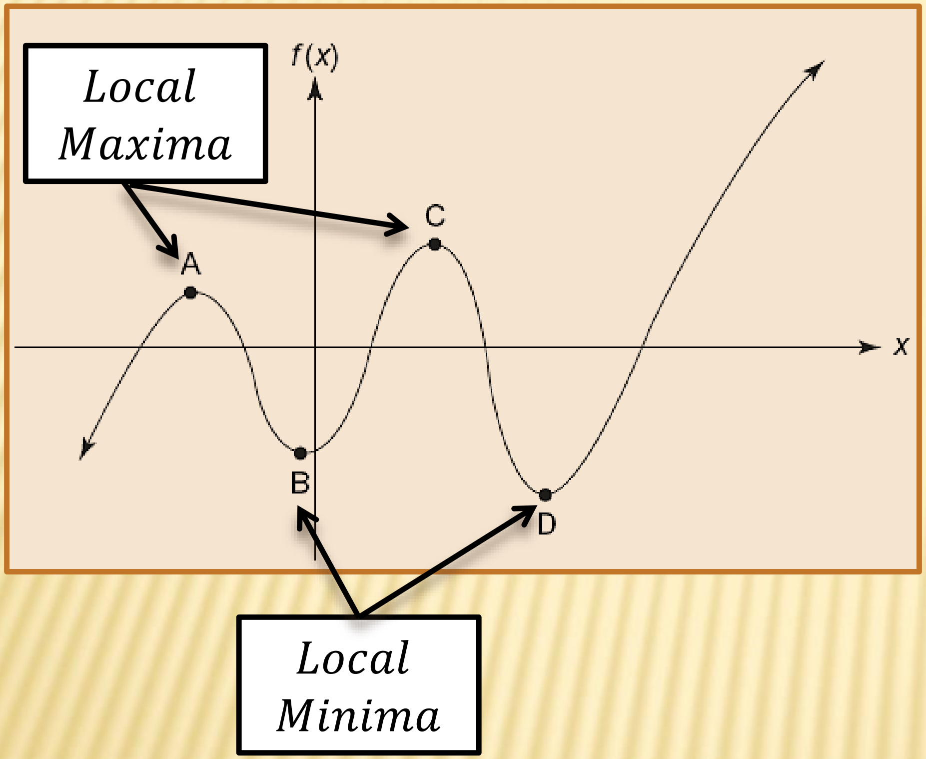
- **Local minimum: a function $f(x)$ of n variables has a local minimum at x^* if $f(x^*)$ is less than or equal to $f(x)$ in a small neighborhood of x^* in the feasible set S.**

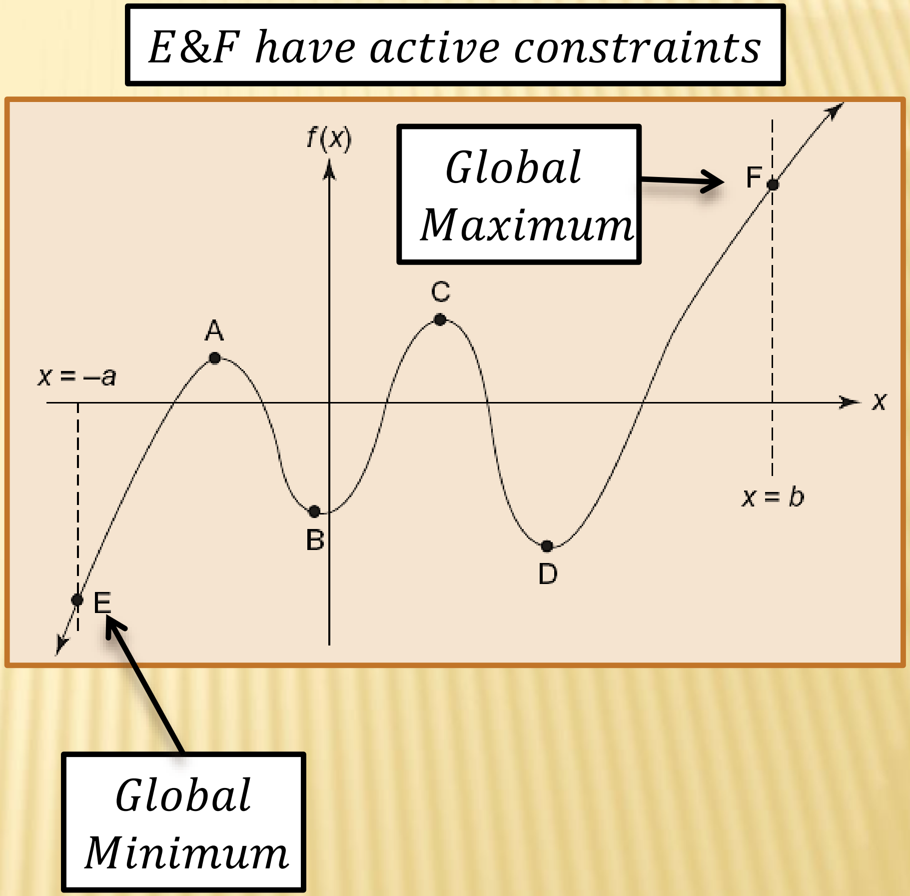
$$f(x^*) \leq f(x) \; \forall \; x \in S$$

$\|x - x^*\| < \delta, \; \delta > 0$ is small value

- **Global & local maxima are defined in a similar manner.**

GLOBAL & LOCAL MINIMA





EXISTENCE OF A MINIMUM

THEOREM 1

Weierstarss Theorem- Existence of a Global Minimum:

If $f(x)$ is continuous on a nonempty feasible set S that is closed and bounded, then $f(x)$ has a global minimum in S

A set S is **closed** if there is **no "< type"** inequality constraints in the formulation of the optimization problem.

A set S is **bounded** if for any point $\boldsymbol{x} \in \boldsymbol{S}, \boldsymbol{x}^T\boldsymbol{x} < \boldsymbol{c}$, c is a finite number.

Karl Wilhelm Weierstarss (1815 – 1897): German mathematician "father of modern analysis"

EXISTENCE OF A MINIMUM

Example 10: Check the existence of a global minimum for the following functions

a. $f(x) = -1/x$ **defined on** $S = \{x \mid 0 < x \leq 1\}$

b. $f(x) = -1/x$ **defined on** $S = \{x \mid 0 \leq x \leq 1\}$

c. $f(x) = x^2$ **defined on** $S = \{x \mid -10 \leq x \leq 10\}$

d. $f(x) = (1/3)\, x^2 + cos x$ **defined on** $S = \{x \mid -\infty < x < \infty\}$

Solution

a. The feasible set S is **not closed** →
Weierstarss Theorem is **not satisfied**, **no global minimum**

b. The function is **not continuous** **at** x**=0** →
Weierstarss Theorem is **not satisfied**, **no global minimum**

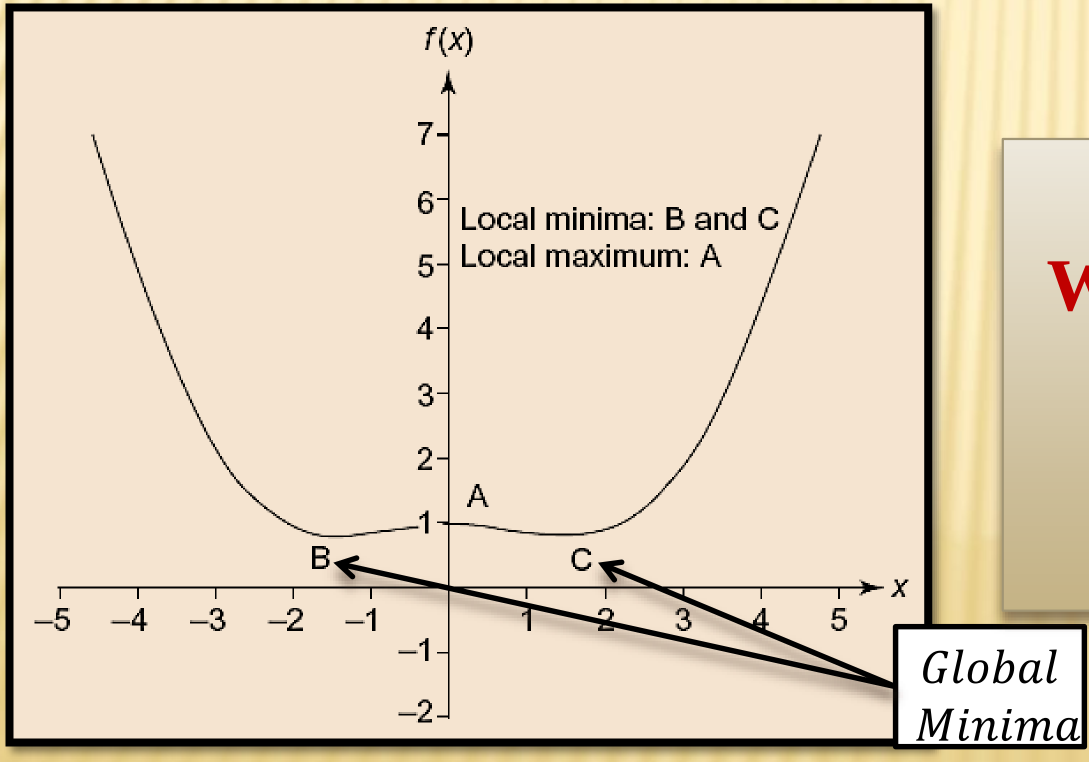
EXISTENCE OF A MINIMUM

c. The feasible set ***S*** is **closed &** the function is **continuous at all *x***

⟶ Weierstarss Theorem is **satisfied**, **global minimum**

d. The feasible set ***S*** is **not closed & unbounded** ⟶

Weierstarss Theorem is **not satisfied**, **no global minimum**



Conclusion

Weierstarss Theorem is not “if-and-only if” theorem

NECESSARY & SUFFICIENT CONDITIONS

Necessary Conditions

The conditions that must be satisfied at the optimum point

Sufficient Conditions

A candidate point satisfies the sufficient conditions is indeed an optimum point

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

Single-Variable

Taylor's Expansion

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}^*) + \boldsymbol{f}'(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*)$$
$$+ \frac{1}{2}\boldsymbol{f}''(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*)^2 + \boldsymbol{R};$$

$\boldsymbol{R}$: is small remainder term

If $\boldsymbol{x}^*$ is a local minimum; a change in the function for any move in a small neighborhood of $\boldsymbol{x}^*$ must be non-negative;

$$\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}^*) = \Delta\boldsymbol{f} \geq \mathbf{0};$$

$$\Delta\boldsymbol{f} = \boldsymbol{f}'(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + \frac{1}{2}\boldsymbol{f}''(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*)^2 + \boldsymbol{R} \geq \mathbf{0};$$

$(\boldsymbol{x} - \boldsymbol{x}^*)$ is very small; first term dominates other terms;

$$\boldsymbol{f}'(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) \geq \mathbf{0} \ \boldsymbol{if}$$

$$\boldsymbol{f}'(\boldsymbol{x}^*) = \mathbf{0}$$ **(Necessary condition)**

Brook Taylor (1685 – 1731): English mathematician

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

$$\boldsymbol{because\ f'(x^*) = 0}$$

$$\Delta f = \frac{1}{2} f''(x^*)(x - x^*)^2 + R \geq 0;$$

second term dominates other terms;

$$f''(x^*)(x - x^*)^2 \geq 0 \ if$$

$f''(x^*) > 0$ **(Sufficient condition)**

$f''(x^*) = 0$ (Evaluate higher-order derivatives)

Example 11: Find the local minimum using necessary/sufficient conditions:

a. $f(x) = x^2 - 4x + 4$

b. $f(x) = x^3 - x^2 - 4x + 4$

c. $f(x) = x^4$

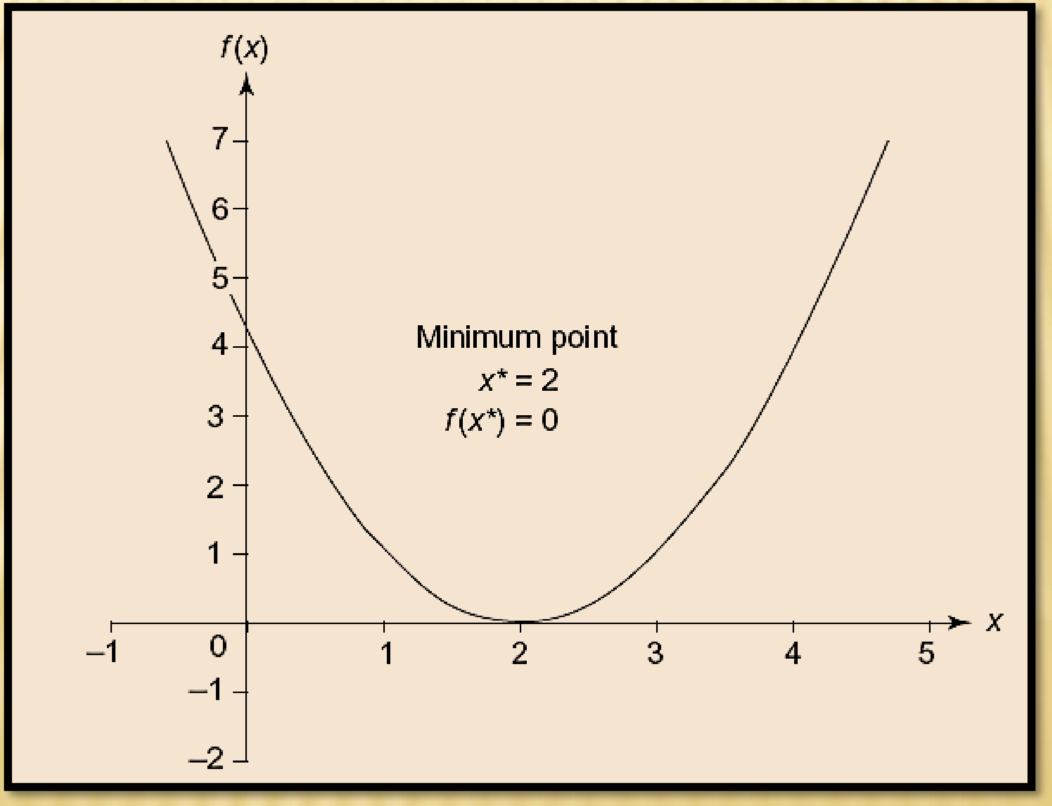
OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

Solution

Candidate Optimum Point

a. $f(x) = x^2 - 4x + 4 \longrightarrow f'(x) = 2x - 4 = 0;\ x^* = 2 \longrightarrow$

$f''(x) = 2 > 0\ \forall\ x \in S;$ ***x^* is indeed local minimum***



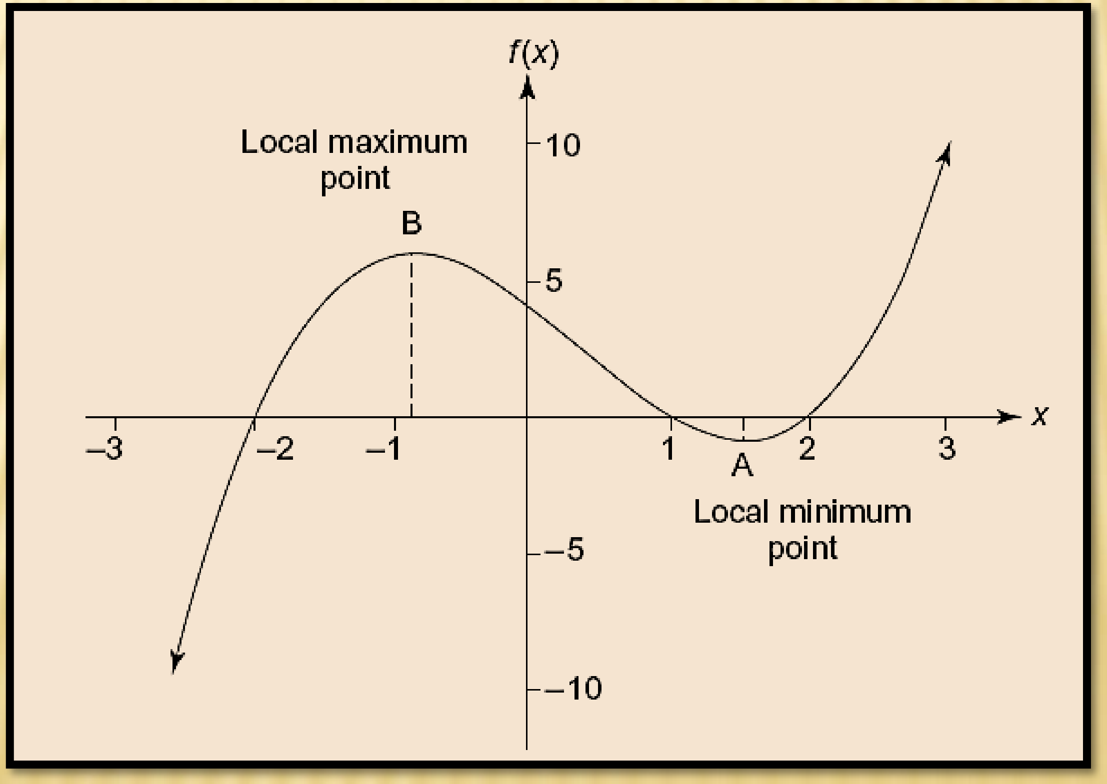
OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

c. $f(x) = x^3 - x^2 - 4x + 4$ ⟶ $f'(x) = 3x^2 - 2x - 4 = 0$

$x_1{}^* = 1.535$ (A), $x_2{}^* = -0.8685$ (B) ⟶ $f''(x) = 6x - 2$

$f''(1.535) = 7.211 > 0$, $f''(-0.8685) = -7.211 < 0$

$x_1{}^*$ is indeed local minimum



OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

c. $f(x) = x^4$ → $f'(x) = 4x^3$=0; x^*=0 →

$f''(x) = 12x^2; f''(x^*) = 0$ → $f'''(x) = 24x; f'''(x^*) = 0$

→ $f''''(x) = 24; f''''(x^*) > 0 \; \forall \; x \in S$ →

x^* *is indeed local minimum*

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

Multi-Variables Optimality Conditions

Taylor's Expansion

$$\boldsymbol{f(x_1, x_2) = f(x_1{}^*, x_2{}^*) + \frac{\partial f}{\partial x_1}(x_1 - x_1{}^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2{}^*) + \frac{1}{2}[\frac{\partial^2 f}{\partial x_1{}^2}(x_1 - x_1{}^*)^2}$$

$$\boldsymbol{+\, 2\,\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1{}^*)(x_2 - x_2{}^*) + \frac{\partial^2 f}{\partial x_2{}^2}(x_2 - x_2{}^*)^2] + R;}$$

R: is small remainder term

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

Taylor's Expansion

$$f(\mathrm{x}) = f(\mathrm{x}^*) + \nabla f^T(\mathrm{x}-\mathrm{x}^*) + \frac{1}{2}(\mathrm{x}-\mathrm{x}^*)^T H(\mathrm{x}-\mathrm{x}^*) + R;$$

$$\mathrm{x}=[x_1\ x_2 \cdots\ x_n]^T,\ column\ vector;$$

$$\nabla f = [\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\cdots\frac{\partial f}{\partial x_n}]^T;$$

(gradient vector-first order partial derivative)

$$H = \frac{\partial^2 f}{\partial \mathrm{x}\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

(Hessian Matrix-second order partial derivative)

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

If $\mathbf{x}^*$ is a local minimum; a change in the function for any move in a small neighborhood of $\mathbf{x}^*$ must be non-negative;

$$\nabla f(\mathbf{x}^*) = \mathbf{0} \text{ (Necessary condition);}$$

$$(\mathbf{x}-\mathbf{x}^*)^T H(\mathbf{x}-\mathbf{x}^*) > \mathbf{0} \text{ (Sufficient condition)}$$

$$\underline{\textit{If } H \textit{ is positive definite matrix } \forall(\mathbf{x}-\mathbf{x}^*) \neq \mathbf{0}}$$

Ludwig Otto Hesse (1811 –1874): German mathematician

NECESSARY & SUFFICIENT CONDITIONS MATRIX FORM

THEOREM 2

$$If\ F(\mathrm{x}) = \mathrm{x}^T A\ \mathrm{x}$$

$$is\ a\ quadratic\ Form\ function, it\ can\ be$$

1. **Positive Definite:** $F(\mathrm{x}) > 0\ \forall\ \mathrm{x} \neq 0$; **A is called Positive Definite Matrix**
2. **Positive Semidefinite:** $F(\mathrm{x}) \geq 0\ \forall\ \mathrm{x} \neq 0$; **$A$ is called Positive Semidefinite Matrix**
3. **Negative Definite:** $F(\mathrm{x}) < 0\ \forall\ \mathrm{x} \neq 0$; **$A$ is called Negative Definite Matrix**
4. **Negative Semidefinite:** $F(\mathrm{x}) \leq 0\ \forall\ \mathrm{x} \neq 0$; **$A$ is called Negative Semidefinite Matrix**
5. **Indefinite:** $F(\mathrm{x}) > 0\ \ for\ some\ values\ of\ \mathrm{x}\ \&\ F(\mathrm{x}) < 0\ for\ some\ others$; **A is called Indefinite Matrix**

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

THEOREM 3

Eigenvalue Check for the Form of a Matrix Let λ_i, $i = 1$ to n be the eigenvalues of a symmetric $n \times n$ matrix **A** associated with the quadratic form $F(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ (since **A** is symmetric, all eigenvalues are real). The following results can be stated regarding the quadratic form $F(\mathbf{x})$ or the matrix **A**:

1. $F(\mathbf{x})$ is *positive definite if and only if* all eigenvalues of **A** are strictly positive; i.e., $\lambda_i > 0$, $i = 1$ to n.
2. $F(\mathbf{x})$ is *positive semidefinite if and only if* all eigenvalues of **A** are non-negative; i.e., $\lambda_i \geq 0$, $i = 1$ to n (note that at least one eigenvalue must be zero for it to be called positive semidefinite).
3. $F(\mathbf{x})$ is *negative definite if and only if* all eigenvalues of **A** are strictly negative; i.e., $\lambda_i < 0$, $i = 1$ to n.
4. $F(\mathbf{x})$ is *negative semidefinite if and only if* all eigenvalues of **A** are nonpositive; i.e., $\lambda_i \leq 0$, $i = 1$ to n (note that at least one eigenvalue must be zero for it to be called negative semidefinite).
5. $F(\mathbf{x})$ is *indefinite if* some $\lambda_i < 0$ and some other $\lambda_j > 0$.

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

THEOREM 4

Check for the Form of a Matrix Using Principal Minors Let M_k be the kth leading principal minor of the $n \times n$ symmetric matrix **A** defined as the determinant of a $k \times k$ submatrix obtained by deleting the last $(n-k)$ rows and columns of **A** (Section A.3). Assume that *no two consecutive principal minors* are zero. Then

1. **A** is positive definite if and only if all $M_k > 0$, $k = 1$ to n.
2. **A** is positive semidefinite if and only if $M_k > 0$, $k = 1$ to r, where $r < n$ is the rank of **A** (refer to Section A.4 for a definition of the rank of a matrix).
3. **A** is negative definite if and only if $M_k < 0$ for k odd and $M_k > 0$ for k even, $k = 1$ to n.
4. **A** is negative semidefinite if and only if $M_k < 0$ for k odd and $M_k > 0$ for k even, $k = 1$ to $r < n$.
5. **A** is indefinite if it does not satisfy any of the preceding criteria.

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

- **How determine matrix form?**
  - **Quadratic Form**
  - **Eigenvalues**
  - **Principal Minors**

Example 12: Determine the matrix form of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

- **Quadratic Form:**

The quadratic form associated with the matrix is:

$$F(\mathrm{x}) = \mathrm{x}^T A\, \mathrm{x} = [x_1 \quad x_2 \quad x_3]^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (2{x_1}^2 + 4{x_2}^2 + 3{x_3}^2) > 0 \ \forall\ \mathrm{x} \neq 0$$

The matrix *A* is Positive Definite Matrix

- **Eigenvalues:**

For a given matrix *A*, the eigenvalue problem is defined as:

$$A\,\mathrm{x} = \lambda \mathrm{x}\ ; |(A - \lambda \mathrm{I})| = 0$$

$$\left| \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\lambda_1 = 2, \lambda_2 = 3\ , \lambda_3 = 4$$

Since all eigenvalues > 0

The matrix *A* is Positive Definite Matrix

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

- **Principal Minor:**

$$M_1 = 2 > 0,\ M_2 = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 > 0,\ M_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24 > 0$$

The matrix *A* is Positive Definite Matrix

Example 13: Find the local minimum using necessary/sufficient conditions:

$$f(\mathrm{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 + x_2 + 8$$

Solution

$$f(\mathrm{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 + x_2 + 8 \longrightarrow$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2x_2 - 2 \\ 2x_1 + 4x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ x_1^* = 2.5,\ x_2^* = -1.5$$

OPTIMALITY CONDITIONS FOR UNCONSTRAINED PROBLEMS

$$\mathbf{x}^* = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix}$$ **(Necessary condition)**

$$H(2.5,-1.5) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

Using one method to determine the matrix form, e.g., eigenvalues

→ $\lambda_1 = 0.7639 > 0, , \lambda_2 = 5.2361 > 0$ →

The matrix <u>*H* is Positive Definite Matrix</u> (Sufficient condition)

→ $\mathbf{x}^*$ ***is indeed local minimum***

Thank You for Your Attention