Elective 2
Optimal Control Systems (ACE 326)

Lecture 5- Calculus-Based Optimization and Basic Concepts --Cont.
Ref. 1: Chapter 4

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OUTLINES

- Optimality conditions for constrained problems
  - Equality constraints
  - Inequality constraints

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

THEOREM 5

Lagrange Multiplier Theorem for equality constraints

Minimize $f(\mathbf{x})$

subject to equality constraints

$$h_i(\mathbf{x}) = 0, i = 1{:}p$$

If $L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x})$ and $\mathbf{x}^*$ is a local minimum, then

$$\nabla L(\mathbf{x}^*, \mathbf{v}^*) = 0 \text{ (Necessary condition);}$$

$$d^T \nabla^2 L(\mathbf{x}^*, \mathbf{v}^*) d > 0, \text{ (Sufficient condition)}$$

$$\underline{If\ \nabla^2 L(\mathbf{x}^*, \mathbf{v}^*) is\ positive\ definite\ matrix\ \forall d \neq 0}$$

Joseph-Louis Lagrange (1736 –1813): an Italian mathematician and astronomer

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

Proof:

Let $\boldsymbol{f}(\mathbf{x}) = \boldsymbol{f}(x_1, x_2)$; $\boldsymbol{h}(\mathbf{x}) = \boldsymbol{h}(x_1, x_2)$ **and** $x_2 = \boldsymbol{\varphi}(x_1)$

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1} \text{ and } \frac{dh}{dx_1} = \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_2}\frac{dx_2}{dx_1}$$

By applying necessary condition

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{d\varphi}{dx_1} = 0$$

Differentiate $\boldsymbol{h}(x_1, x_2)$ **at the point** $(x_1{}^*, x_2{}^*)$

$$\frac{dh}{dx_1} = \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_2}\frac{dx_2}{dx_1} = \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_2}\frac{d\varphi}{dx_1} = 0$$

$$\frac{d\varphi}{dx_1} = -\frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

Define a parameter $v = -\frac{\partial f/\partial x_2}{\partial h/\partial x_2}$ **(Lagrange multiplier)**

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

$$\frac{\partial f}{\partial x_1} + v\frac{\partial h}{\partial x_1} = 0 \ \& \ \frac{\partial f}{\partial x_2} + v\frac{\partial h}{\partial x_2} = 0$$

These are the necessary conditions for equality constraints

Define $L(x_1{}^*, x_2{}^*, v^*) = f(x_1{}^*, x_2{}^*) + vh(x_1{}^*, x_2{}^*)$

The necessary conditions can be rewritten as:

$$\frac{\partial L}{\partial x_1}=0 \ \& \ \frac{\partial L}{\partial x_2}=0 \ \& \ \frac{\partial L}{\partial v}=0$$

OR

$$\nabla L(x_1{}^*, x_2{}^*, v) = 0$$

End of proof.

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

Remarks:

$$\nabla \boldsymbol{f}(x_1{}^*, x_2{}^*) + v^* \nabla \boldsymbol{h}(x_1{}^*, x_2{}^*) = \mathbf{0}$$

Then

$$\nabla \boldsymbol{f}(x_1{}^*, x_2{}^*) = -v^* \nabla \boldsymbol{h}(x_1{}^*, x_2{}^*)$$

1. **The geometrical meaning of the necessary conditions:**

At the candidate optimum point, the gradients of both the cost function and equality constraints are along the same line and proportional to each other, v^* is the proportionality constant.

2. **The Lagrange multiplier is sign free.**

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

Example 14: Minimize

$$f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

Subject to an equality constraint

$$h(x_1, x_2) = x_1 + x_2 - 2 = 0$$

Solution

$$L(x_1, x_2, v) = f(x_1, x_2) + vh(x_1, x_2)$$
$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + v(x_1 + x_2 - 2)$$

Necessary condition: $\nabla \boldsymbol{L}(x_1{}^*, x_2{}^*, v^*) = 0$

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + v = 0, \quad \frac{\partial L}{\partial x_2} = 2(x_2 - 1.5) + v = 0,$$

$$\frac{\partial L}{\partial v} = x_1 + x_2 - 2 = 0$$

→ $x_1{}^* = 1, x_2{}^* = 1, and v^* = 1$

Candidate optimum point

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

To check that the candidate optimum point verifies the geometrical meaning of the necessary condition,

$$\nabla \boldsymbol{f}({x_1}^*, {x_2}^*) = \begin{bmatrix} -\mathbf{1} \\ -\mathbf{1} \end{bmatrix}, \nabla \boldsymbol{h}({x_1}^*, {x_2}^*) = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

$$\nabla \boldsymbol{f}({x_1}^*, {x_2}^*) = -\nabla \boldsymbol{h}({x_1}^*, {x_2}^*)$$

Then, at the candidate optimum point (1,1), the gradients of both the cost function and equality constraints are along the same line and proportional to each other, $v^* = 1$ is the proportionality constant.

OPTIMALITY CONDITIONS FOR INEQUALITY CONSTRAINED PROBLEMS

THEOREM 6

Lagrange Multiplier Theorem for inequality constraints

Minimize $f(\mathbf{x})$

subject to inequality constraints

$$g_j(\mathbf{x}) \leq 0, j = 1: m$$

If $L(\mathbf{x}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{j=1}^{m} u_j(g_j(\mathbf{x}) + {s_j}^2)$ **and** $\mathbf{x}^*$**is a local minimum, then**

$$\nabla L(\mathbf{x}^*, \mathbf{u}^*, \mathbf{s}^*) = \mathbf{0}$$ **(Necessary condition);**

$$d^T \nabla^2 L(\mathbf{x}^*, \mathbf{u}^*, \mathbf{s}^*) d > 0,$$ **(Sufficient condition)**

$\underline{If\ \nabla^2 L(\mathbf{x}^*, \mathbf{u}^*, \mathbf{s}^*) is\ positive\ definite\ matrix\ \forall d \neq 0}$

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

Proof:

Because $g_j(\mathrm{x}) \leq 0, j = 1{:}m$, **there exists a variable** $s_j \geq 0$, where

$$s_j{}^2 \geq 0, j = 1{:}m$$

$$g_j(\mathrm{x}) + s_j{}^2 = 0$$

Define a parameter $u_j \geq 0$ **(Lagrange multiplier)**

The necessary conditions for inequality constraints

Define $L(\mathrm{x}^*, \mathrm{u}^*, \mathrm{s}^*) = f(\mathrm{x}^*) + \mathrm{u}^T(g(\mathrm{x}^*) + s^2)$

The necessary conditions can be rewritten as:

$$\frac{\partial L}{\partial x_k}=0 \ \& \ \frac{\partial L}{\partial u_j}=0 \ \& \ \frac{\partial L}{\partial s_j}=0$$

OR

$$\nabla L(\mathrm{x}^*, \mathrm{u}^*, \mathrm{s}^*) = 0$$

End of proof.

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

Remarks:

1. **The Lagrange multiplier for "$\leq$" inequality constraint must be nonnegative.**

OPTIMALITY CONDITIONS FOR INEQUALITY CONSTRAINED PROBLEMS

Example 15: Minimize

$$f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

Subject to an equality constraint

$$g(x_1, x_2) = x_1 + x_2 - 2 \leq 0$$

Solution

$$L(x_1, x_2, u, s) = f(x_1, x_2) + u(g(x_1, x_2) + s^2)$$
$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + u(x_1 + x_2 - 2 + s^2)$$

Necessary condition: $\boldsymbol{\nabla L}(x_1{}^*, x_2{}^*, v) = 0$

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + u = 0, \quad \frac{\partial L}{\partial x_2} = 2(x_2 - 1.5) + u = 0,$$

$$\frac{\partial L}{\partial u} = x_1 + x_2 - 2 + s^2 = 0, \quad \frac{\partial L}{\partial s} = 2us = 0$$

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

$When\ s^* = 0$ → $x_1{}^* = 1, x_2{}^* = 1, and\ u^* = 1$

$Candidate\ optimum\ point$

$\boldsymbol{s_j \geq 0\ is\ verified}$

$When\ u^* = 0$ → $x_1{}^* = 1.5, x_2{}^* = 1.5, and\ s^{*2} = -1$

$\boldsymbol{s_j < 0\ is\ not\ verified}$

OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINED PROBLEMS

To check that the candidate optimum point verifies the geometrical meaning of the necessary condition,

$$\nabla \boldsymbol{f}({x_1}^*, {x_2}^*) = \begin{bmatrix} -\mathbf{1} \\ -\mathbf{1} \end{bmatrix}, \nabla \boldsymbol{g}({x_1}^*, {x_2}^*) = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

$$\nabla \boldsymbol{f}({x_1}^*, {x_2}^*) = -\nabla \boldsymbol{g}({x_1}^*, {x_2}^*)$$

Then, at the candidate optimum point (1,1), the gradients of both the cost function and inequality constraints are along the same line and proportional to each other, $u^* = 1$ is the proportionality constant.

OPTIMALITY CONDITIONS FOR GENERAL CONSTRAINED PROBLEMS

THEOREM 7

Karush-Kuhn-Tucker (KKT) necessary conditions

Minimize $f(\mathbf{x})$
subject to equality and inequality constraints

$$h_i(\mathbf{x}) = 0, i = 1{:}\,p$$
$$g_j(\mathbf{x}) \le 0, j = 1{:}\,m$$

If $L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j (g_j(\mathbf{x}) + s_j{}^2)$ and $\mathbf{x}^*$is a local minimum, then

$$\nabla L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{s}^*) = 0$$ (Necessary condition);

$$d^T \nabla^2 L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{s}^*) d > 0,$$ (Sufficient condition)

$$\underline{If\ \nabla^2 L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{s}^*) is\ positive\ definite\ matrix\ \forall d \neq 0}$$

William Karush (1917 – 1997) an American mathematician
Harold William Kuhn (1925 –2014) an American mathematician
Albert William Tucker (1905 – 1995) a Canadian mathematician

OPTIMALITY CONDITIONS FOR GENERAL CONSTRAINED PROBLEMS

Example 16: Minimize the following function using KKT conditions:

$$f(x_1, x_2) = {x_1}^2 + {x_2}^2 - 3x_1\, x_2$$

Subject to an equality constraint

$$g(x_1, x_2) = {x_1}^2 + {x_2}^2 - 6 \leq 0$$

Solution

$$L(x_1, x_2, u, s) = f(x_1, x_2) + u(g(x_1, x_2) + s^2)$$
$$= {x_1}^2 + {x_2}^2 - 3x_1\, x_2 + u({x_1}^2 + {x_2}^2 - 6 + s^2)$$

Necessary condition: $\nabla \boldsymbol{L}({x_1}^*, {x_2}^*, v) = 0$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 3x_2 + 2ux_1 = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 - 3x_1 + 2ux_2 = 0,$$

$$\frac{\partial L}{\partial u} = {x_1}^2 + {x_2}^2 - 2 + s^2 = 0, \qquad \frac{\partial L}{\partial s} = 2us = 0$$

OPTIMALITY CONDITIONS FOR GENERAL CONSTRAINED PROBLEMS

Case 1: $u^* = 0$ → $x_1{}^* = 0, x_2{}^* = 0, and\ s^2 = 6 \geq 0$

Case 2: $s^* = 0$ → $x_1{}^* = x_2{}^* = \sqrt{3}\ and\ u^* = 1/2 > 0$

$x_1{}^* = x_2{}^* = -\sqrt{3}\ and\ u^* = 1/2 > 0$

Candidate optimum points

$x_1{}^* = -x_2{}^* = \sqrt{3}\ and\ u^* = -5/2$

$x_1{}^* = -x_2{}^* = -\sqrt{3}\ and\ u^* = -5/2$

Case 3: $u^* = 0,\ s^* = 0$ → $x_1{}^* = 0, x_2{}^* = 1, and\ s^2 = 6 \neq 0$

OPTIMALITY CONDITIONS FOR GENERAL CONSTRAINED PROBLEMS

Remarks about KKT necessary conditions:

1. The conditions can be used to check whether a given point is a candidate minimum.
2. Several cases must be considered and solved. Each case can provide multiple solutions.
3. For each solution case, remember to
   a. Check all inequality constraints for feasibility ($\boldsymbol{s}^{*2} \geq 0$)
   b. Calculate all of the Lagrange multipliers.
   c. Ensure that the Lagrange multipliers for all of the inequality constraints are nonnegative ($\boldsymbol{u_j}^{*} \geq 0$).
4. Number of cases=2 Number of inequalities.

Thank You for Your Attention