Elective 2
Optimal Control Systems
(ACE 326)

Lecture 6- Optimal Control Systems
Ref. 2: Chapters 1&2

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OUTLINES

- Historical Tour
- Modern Control Theory Vs. Conventional Control Theory
- Optimal Control System

HISTORICAL TOUR

1. Johannes Bernoulli in 1699 posed the problem "finding the path of quickest descent between two points not in the same horizontal or vertical line".
2. Leonhard Euler in 1750 solved the problem and is founder of calculus of variations in 1766.
3. Joseph-Louis Lagrange in 1755 found necessary condition "Euler - Lagrange equation".
4. Andrien Marie Legendre in 1786 found sufficient condition.
5. Carl Gustav Jacob Jacobi in 1836 came up with a more rigorous analysis of the sufficient conditions "Legendre-Jacobi condition".

Johannes Bernoulli (1667-1748): a Swiss mathematician
Leonhard Euler (1707 – 1783): a Swiss mathematician, physicist, astronomer and engineer
Andrien Marie Legendre (1752-1833): a French mathematician
Carl Gustav Jacob Jacobi (1804-1851): a German mathematician

HISTORICAL TOUR

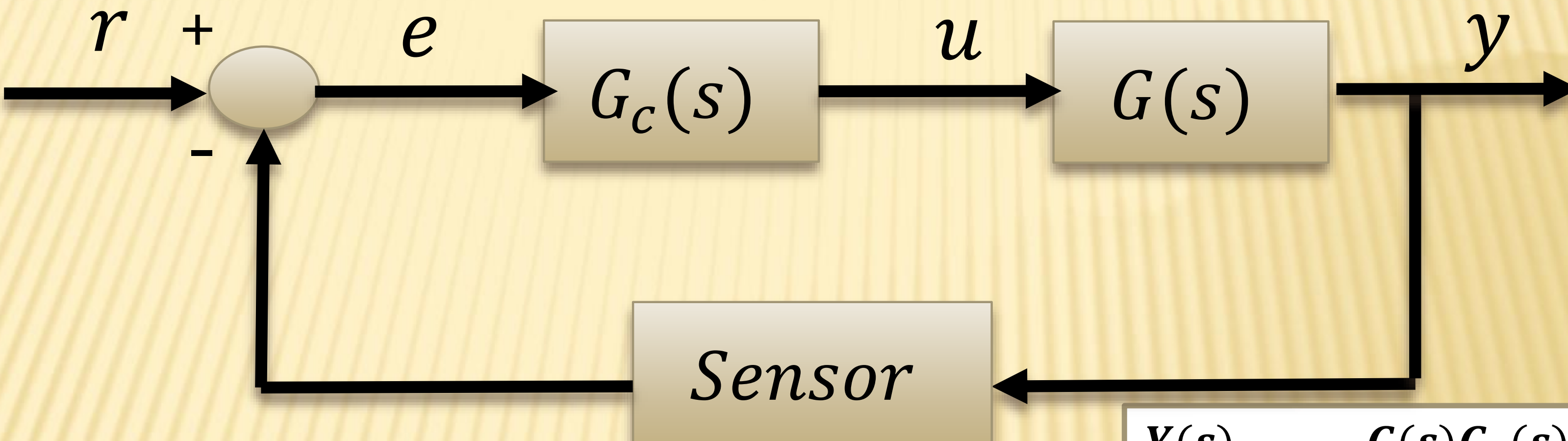
6. William Rowan Hamilton in 1838 did some remarkable work on mechanics, motion of a particle in space "Hamilton-Jacobi equation".
7. N. Wiener developed optimal control for weapon fire during World War II (1940-1945) .
8. L. S. Pontryagin in 1956 presented "maximum principle "based on work of Hamilton function.
9. R. Bellman in 1957 introduced "dynamic programming "to solve discrete time optimal control problems "Hamilton-Jacobi-Bellman approach".
10. R. E. Kalman in 1960 provided linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) theory to design optimal feedback controls based on work of Pontryagin.

William Rowan Hamilton (1788-1856) an Irish mathematician
N. Wiener (1894 – 1964) an American mathematician and philosopher
Richard Ernest Bellman(1920-1984) an American applied mathematician
Lev Semyonovich Pontryagin (1908 – 1988): a Soviet mathematician
Rudolf Emil Kalman (1930-2016) a Hungarian-born American electrical engineer, mathematician, and inventor.

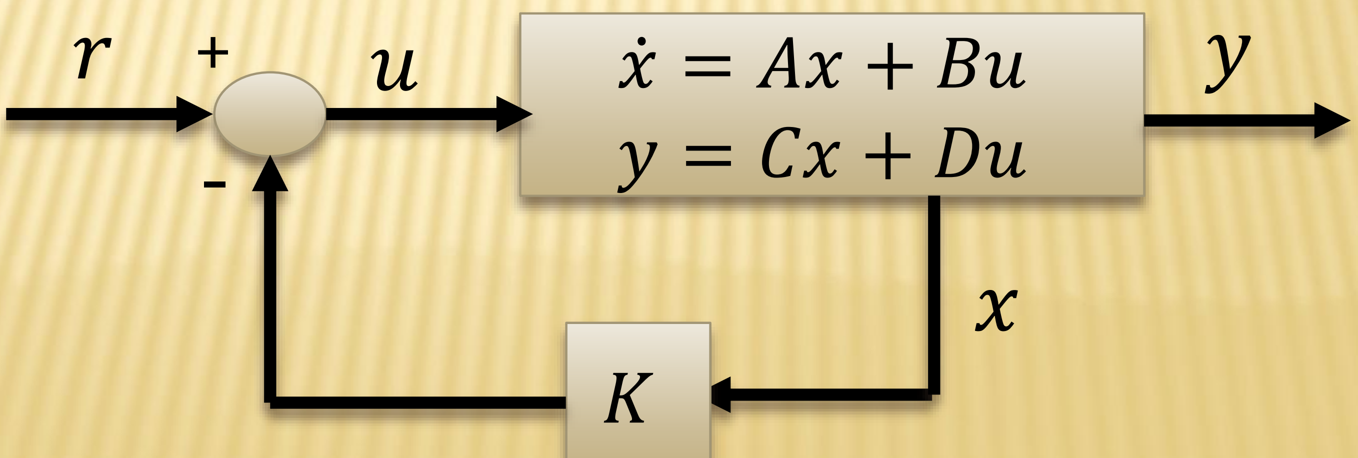
MODERN CONT. VS CONVENTIONAL CONT.

Conventional Control	Modern Control
Frequency domain approach	Time domain approach
Based on Laplace transforms theory	Based on state variable representation
Applicable to SISO, LTI systems only	Applicable to SISO/MIMO – linear/nonlinear, time-invariant/ time-varying
Initial conditions =Zero	Initial conditions ≠ Zero

MODERN CONT. VS CONVENTIONAL CONT.



$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G(s)G_c(s)}}{\boldsymbol{1+G(s)G_c(s)R(s)}}$$



IMPORTANT IDIOMS

- **Calculus of Variations (CoV):** is the branch of mathematics concerning to find maxima and minima of functionals.
- **Functionals:** function of a function. Let J is a functional dependent on a function $f(x)$; $J = V\big(f(x)\big)$; V are often expressed as definite integrals.
- **Quadratic Form**: is a special nonlinear function having only second-order terms (either the square of a variable or the product of two variables).

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}\, x_i\, x_j = x^T P x$$

$$P = [p_{ij}]_{n\times n} matrix$$

OPTIMAL CONTROL SYSTEM

Optimal Control System

Static

- ✓ Plants under steady state conditions
- ✓ Algebraic equations
- ✓ Calculus, Lagrange multipliers

Linear/nonlinear programming

Dynamic

- ✓ Plants under dynamic conditions
- ✓ Differential/difference equations
- ✓ Calculus of variations, Pontryagin principle, dynamic programming/search techniques

OPTIMAL CONTROL SYSTEM

Performance Index

Classical control design

- ✓ Time response (rise time, settling time, peak overshoot, steady state accuracy)
- ✓ Frequency response (gain/phase margin, bandwidth)

Modern control design

Find u^*

- ✓ Reach a target (follow trajectory)
- ✓ Extremize performance index

OPTIMAL CONTROL SYSTEM

Optimal Control System

Plant

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Performance Index

$$J = \int_{t_0}^{t_f} [x^T Q x + u^T R u] dt$$

Constraints

$$U_- \leq u(t) \leq U_+$$
$$x_- \leq x(t) \leq x_+$$

$\boldsymbol{R}$ is positive definite matrix

$\boldsymbol{Q}$ is positive semidefinite matrix

OPTIMAL CONTROL SYSTEM

THEOREM 8

Euler-Lagrange Multiplier Theorem

Minimize cost function $J = \int_{t_0}^{t_f} [\mathbf{x}^T Q \mathbf{x} + u^T R u] dt$

subject to

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

If $V(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}, t) = \mathbf{x}^T Q \mathbf{x} + u^T R u$,

$h(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}, t) = \dot{\mathbf{x}} - A\mathbf{x} - Bu = 0$

$L(\mathbf{x}, \dot{\mathbf{x}}, u, t) = V(\mathbf{x}, \dot{\mathbf{x}}, u, t) + \lambda^T h(\mathbf{x}, \dot{\mathbf{x}}, u, t)$ and $\mathbf{x}^*$, u^* is a local minimum, then

$$\frac{\partial L}{\partial \mathbf{x}^*} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}^*} = 0; \frac{\partial L}{\partial u^*} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}^*} = 0; \frac{\partial L}{\partial \lambda^*} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}^*} = 0 \text{ ; } \mathbf{x}(0) \text{ given}$$

(Necessary condition)

Leonhard Euler (1707 – 1783): a Swiss mathematician, physicist, astronomer and engineer

OPTIMAL CONTROL SYSTEM

Example 14: **Minimize**

$$J=\int_{0}^{1}\left[x^2(t)+u^2(t)\right]dt$$

Subject to plant equation $\dot{x}(t)=-x(t)+u(t)$

with boundary conditions $x(0)=1; x(1)=0$

Solution

$$L=x^2(t)+u^2(t)+\lambda(\dot{x}(t)+x(t)-u(t))$$

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=2x(t)+\lambda-\dot{\lambda}=0 \qquad (1)$$

OPTIMAL CONTROL SYSTEM

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} = 2u(t) - \lambda = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}} = \dot{x}(t) + x(t) - u(t) = 0 \quad (3)$$

From (2) and (3), $\lambda^*(t) = 2u(t) = 2\dot{x}(t) + 2x(t)$

From (1), $\ddot{x}(t) - 2x(t) = 0$

Candidate optimum design

$$x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}$$

$$u^*(t) = C_1(1-\sqrt{2})e^{-\sqrt{2}t} + C_2(1+\sqrt{2})e^{\sqrt{2}t}$$

$$C_1 = \frac{1}{1-e^{-2\sqrt{2}}}, C_2 = \frac{1}{1-e^{2\sqrt{2}}}$$

OPTIMAL CONTROL SYSTEM

THEOREM 9

Pontryagin Maximum Principle

Minimize cost function $J = \int_{t_0}^{t_f} [\mathrm{x}^T Q \mathrm{x} + u^T R u] dt$

subject to

$$f = \dot{\mathrm{x}} = A\mathrm{x} + Bu$$

If $V(\mathrm{x}, \mathrm{u}, t) = \mathrm{x}^T Q \mathrm{x} + u^T R u,$

$$H(\mathrm{x}, u, t) = V(\mathrm{x}, u, t) + \lambda^T f$$

and x^*, u^* **is a local minimum, then**

$$\frac{\partial H}{\partial {x_i}^*} = -\dot{\lambda}_i; \ \frac{\partial H}{\partial {\lambda_i}^*} = \dot{x}_i; \ \frac{\partial H}{\partial u^*} = 0; \mathrm{x}(0) \text{ given}$$

(Necessary condition)

OPTIMAL CONTROL SYSTEM

Example 15: **Minimize using maximum principle**

$$J = \frac{1}{2}\int_0^2 u^2(t)dt$$

Subject to plant equation

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$

with boundary conditions $\mathbf{x}(0) = [1 \;\; 2]^T; \mathbf{x}(2) = [1 \;\; 0]^T$

Solution

$$V = \frac{1}{2}u^2(t), f_1 = x_2(t), f_2 = u(t)$$

$$H = \frac{1}{2}u^2(t) + \lambda_1 f_1 + \lambda_2 f_2 = \frac{1}{2}u^2(t) + \lambda_1 x_2(t) + \lambda_2 u(t) \quad (1)$$

$$\frac{\partial H}{\partial u} = u(t) + \lambda_2(t) = 0; u(t) = -\lambda_2(t) \quad (2)$$

OPTIMAL CONTROL SYSTEM

From (1) and (3), $\mathrm{H} = \frac{1}{2}\lambda_2{}^2(t) + \lambda_1 x_2(t) - \lambda_2{}^2(t)$

$$\dot{x}_1(t) = \frac{\partial H}{\partial \lambda_1} = x_2(t) \qquad (3)$$

$$\dot{x}_2(t) = \frac{\partial H}{\partial \lambda_2} = -\lambda_2(t) \qquad (4)$$

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x_1} = 0 \qquad (5)$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial x_2} = -\lambda_1(t) \qquad (6)$$

OPTIMAL CONTROL SYSTEM

From (5) and (6) $\lambda_1(t) = C_1, \lambda_2(t) = -C_1 t + C_2$

$$x_2(t) = \frac{C_1}{2}t^2 - C_2 t + C_3$$

$$x_1(t) = \frac{C_1}{6}t^3 - \frac{C_2}{2}t^2 + C_3 t + C_4$$

From boundary conditions

$C_4=1, C_3=2, C_2=4$ *and* $C_1=3$

Candidate *optimum* design

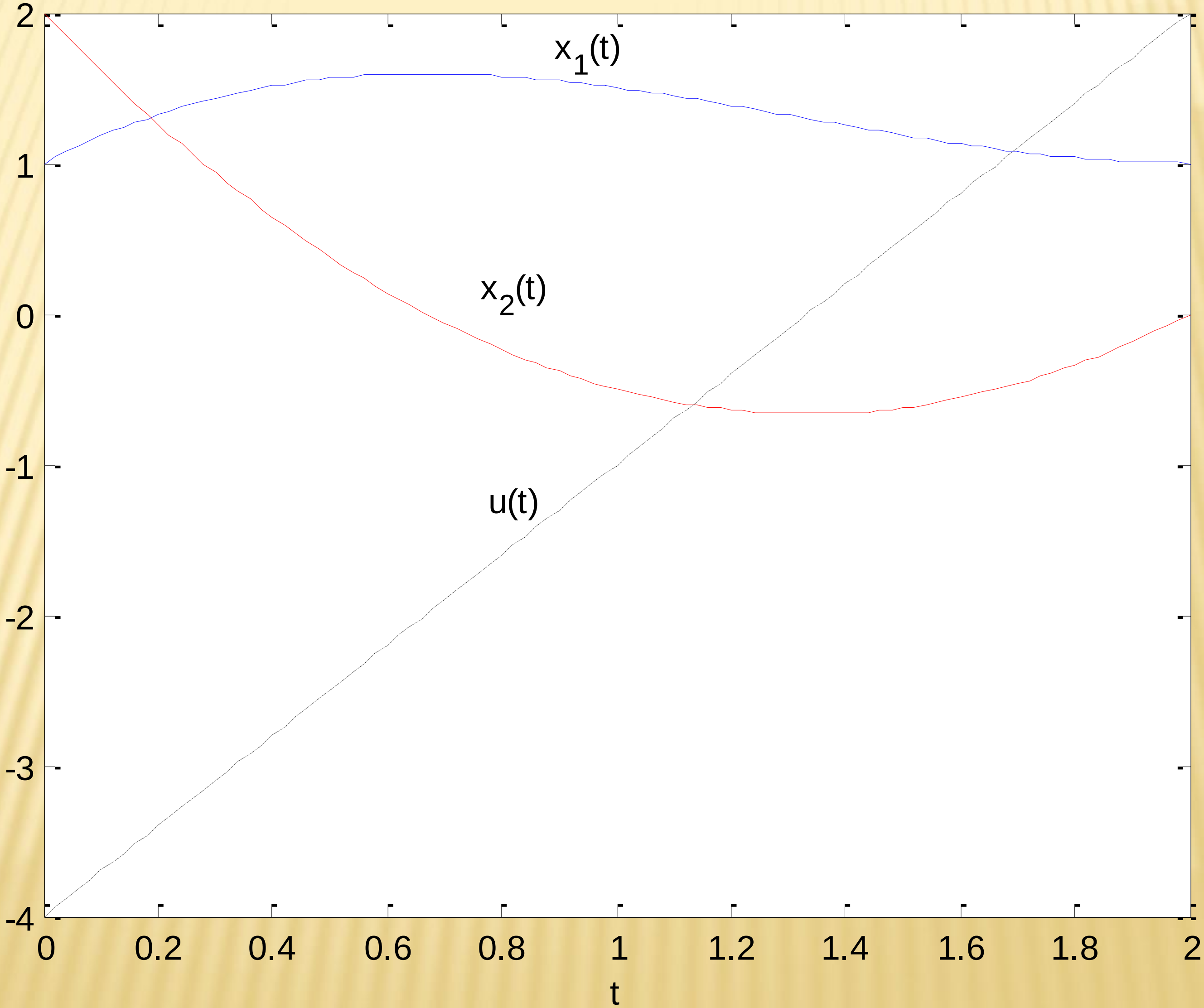
$$x_1^*(t) = \frac{1}{2}t^3 - 2t^2 + 2t + 1$$

$$x_2^*(t) = \frac{3}{2}t^2 - 4t + 2$$

$$\lambda_1^*(t) = 3, \lambda_2^* = -3t + 4$$

$$u^*(t) = 3t - 4$$

OPTIMAL CONTROL SYSTEM



Thank You for Your Attention