Elective 2
Optimal Control Systems
(ACE 326)

Lecture 7- Optimal Control Systems…Cont.

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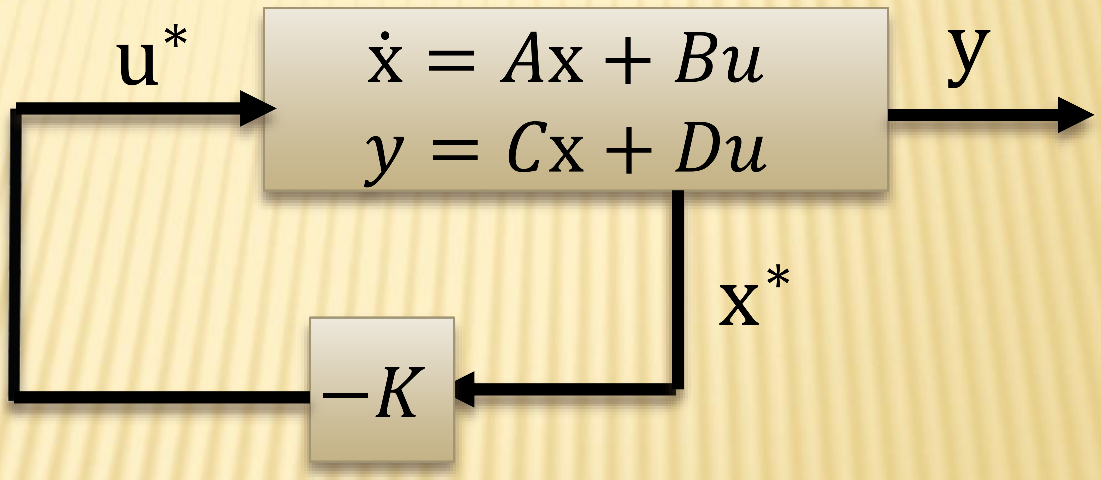
OUTLINES

- Linear Quadratic Regulator



LINEAR QUADRATIC REGULATOR

THEOREM 10

Linear Quadratic Regulator

For a cost function $J = \frac{1}{2}\int_{t_0}^{t_f}[\mathrm{x}^T Q\mathrm{x} + u^T Ru]dt$

subject to

$$f = \dot{\mathrm{x}} = A\mathrm{x} + Bu;\ \mathrm{x}(0) \text{ given}$$

Then $u = -K\mathrm{x} = -R^{-1}B^T P\,\mathrm{x}$

Where $A^T P + PA - PBR^{-1}B^T P + Q = 0$

Rudolf Emil Kalman (1930 – 2016): a Hungarian-born American electrical engineer mathematician, and inventor.

LINEAR QUADRATIC REGULATOR

Proof:

Using maximum principle:

$$H(\mathrm{x},u,t) = V(\mathrm{x},u,t) + \lambda^T f =$$

$$\frac{1}{2}\left(\mathrm{x}^T Q\mathrm{x} + u^T R u\right) + \lambda^T (A\mathrm{x} + Bu)$$

Then $$\frac{\partial H}{\partial u} = Ru(t) + B^T\lambda = 0;\ u(t) = -R^{-1}B^T\lambda$$

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = A\mathrm{x} + Bu$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -(Qx + A^T\lambda)$$

Let $\lambda(t) = P(t)\mathrm{x}(t)$

Riccati Coefficients

LINEAR QUADRATIC REGULATOR

Then $\dot{\lambda}(t) = \dot{P}(t)\mathrm{x}(t) + P(t)\dot{\mathrm{x}}(\mathrm{t}) = \dot{P}(t)\mathrm{x}(t) + P(t)(A\mathrm{x}(\mathrm{t}) - BR^{-1}B^{T}P(t)\mathrm{x}(t))$

$-(Qx + A^{T}\lambda) = -Q\mathrm{x}(t) - A^{T}P(t)\mathrm{x}(t) = \dot{P}(t)\mathrm{x}(t) + P(t)(A\mathrm{x}(\mathrm{t}) - BR^{-1}B^{T}P(t)\mathrm{x}(t))$

This equation is satisfied if we can find $P(t)$ such that:

$$-\dot{P}(t) = P(t)A + A^{T}P(t) + Q - P(t)BR^{-1}B^{T}P(t)$$

Riccati ODE

At infinite time, $t_f = \infty, \dot{P}(t) = 0$,

$$0 = P(t)A + A^{T}P(t) + Q - P(t)BR^{-1}B^{T}P(t)$$

Algebraic Riccati equation

Jacopo Francesco Riccati (1676 – 1754): an Italian mathematician

LINEAR QUADRATIC REGULATOR

➢ **How design LQR?**

Step1: Check the rank of controllability matrix C

$$\boldsymbol{C = [B \quad AB \quad A^2B \ldots A^{n-1}B];}$$

$$\boldsymbol{n = system\ order}$$

✓ $if\ rank(C) = n;\ system\ is\ completely\ state\ controllable$

✓ $if\ rank(C) < n;\ system\ is\ not\ completely\ state\ controllable$

Step2: Find the state feedback gain K

$$\dot{x} = Ax + Bu;\ u = -Kx$$

$$\boldsymbol{min\, J = \frac{1}{2}\int_0^\infty [x^TQx + u^TRu]\ dt}$$

$$K = R^{-1}B^TP;$$

$$PA + A^T P + Q - PBR^{-1}B^TP = 0$$

LINEAR QUADRATIC REGULATOR

Example 16: **Design an LQR**

$$J = \frac{1}{2}\int_0^\infty 2{x_1}^2(t) + 6x_1(t)x_2(t) + 5{x_2}^2(t) + 0.25u^2(t)dt$$

Subject to plant equation

$$\dot{x}_1(t) = x_2(t),\ \ \dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t)$$

with boundary conditions $\mathbf{x}(0) = [2 \ \ -3]^T$

Solution

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

LINEAR QUADRATIC REGULATOR

Step1: Check the rank of controllability matrix *C*

$$\boldsymbol{C} = [\boldsymbol{B} \quad \boldsymbol{AB}] = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$$

$$rank(C) = 2 = n; system\ is\ completely\ state\ controllable$$

Step2: Find the state feedback gain *K*

- $\boldsymbol{Let\ P} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{b} & \boldsymbol{c} \end{bmatrix}$; *Q*= $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}$; *R*= $\frac{\mathbf{1}}{\mathbf{4}}$

- **Solve algebraic Riccati equation**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4 [0 \quad 1] \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

LINEAR QUADRATIC REGULATOR

Simplifying this equation,

$$-4b^2 - 4b + 2 = 0$$

$$a + b - 2c - 4bc + 3 = 0$$

$$2b + 2c - 4c^2 + 5 = 0$$

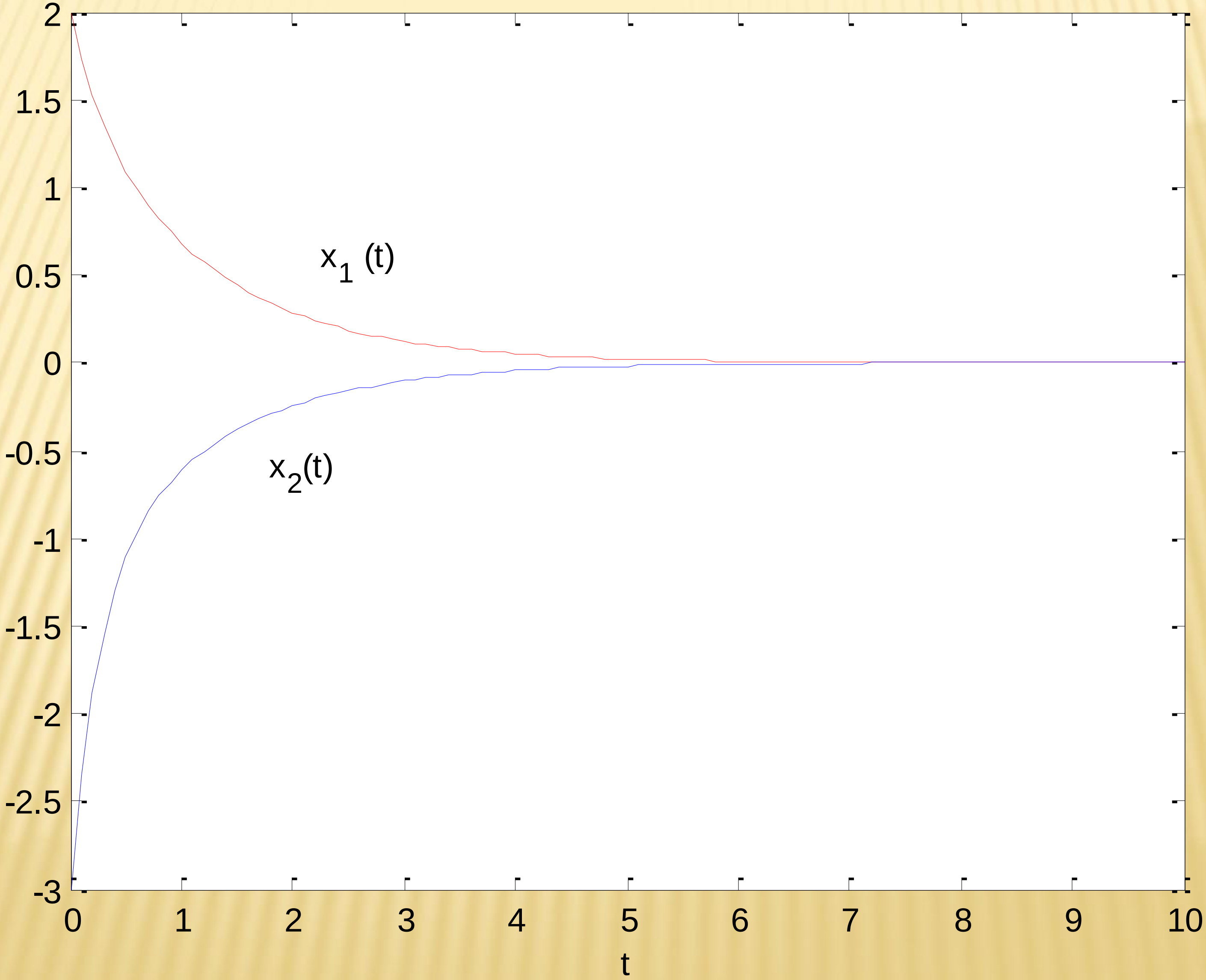
$$P = \begin{bmatrix} 1.7363 & 0.366 \\ 0.366 & 1.4729 \end{bmatrix}$$

$$K = R^{-1}B^T P = 4\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.7363 & 0.366 \\ 0.366 & 1.4729 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4641 & 5.8916 \end{bmatrix}$$

$$u^*(t) = -1.464\, {x_1}^*(t) - 5.8916\, {x_2}^*(t)$$

LINEAR QUADRATIC REGULATOR



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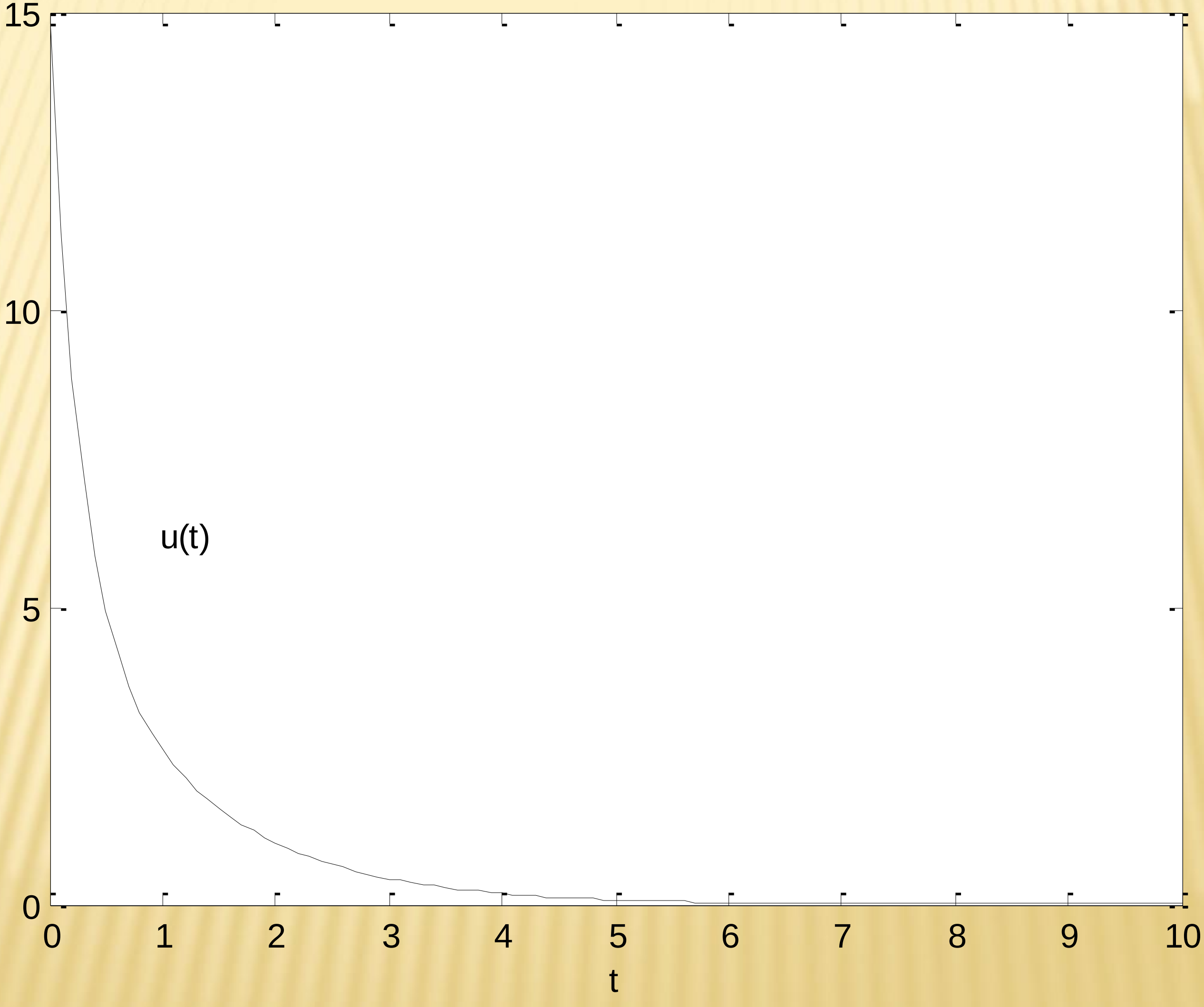


LINEAR QUADRATIC REGULATOR

Remarks about LQR weights:

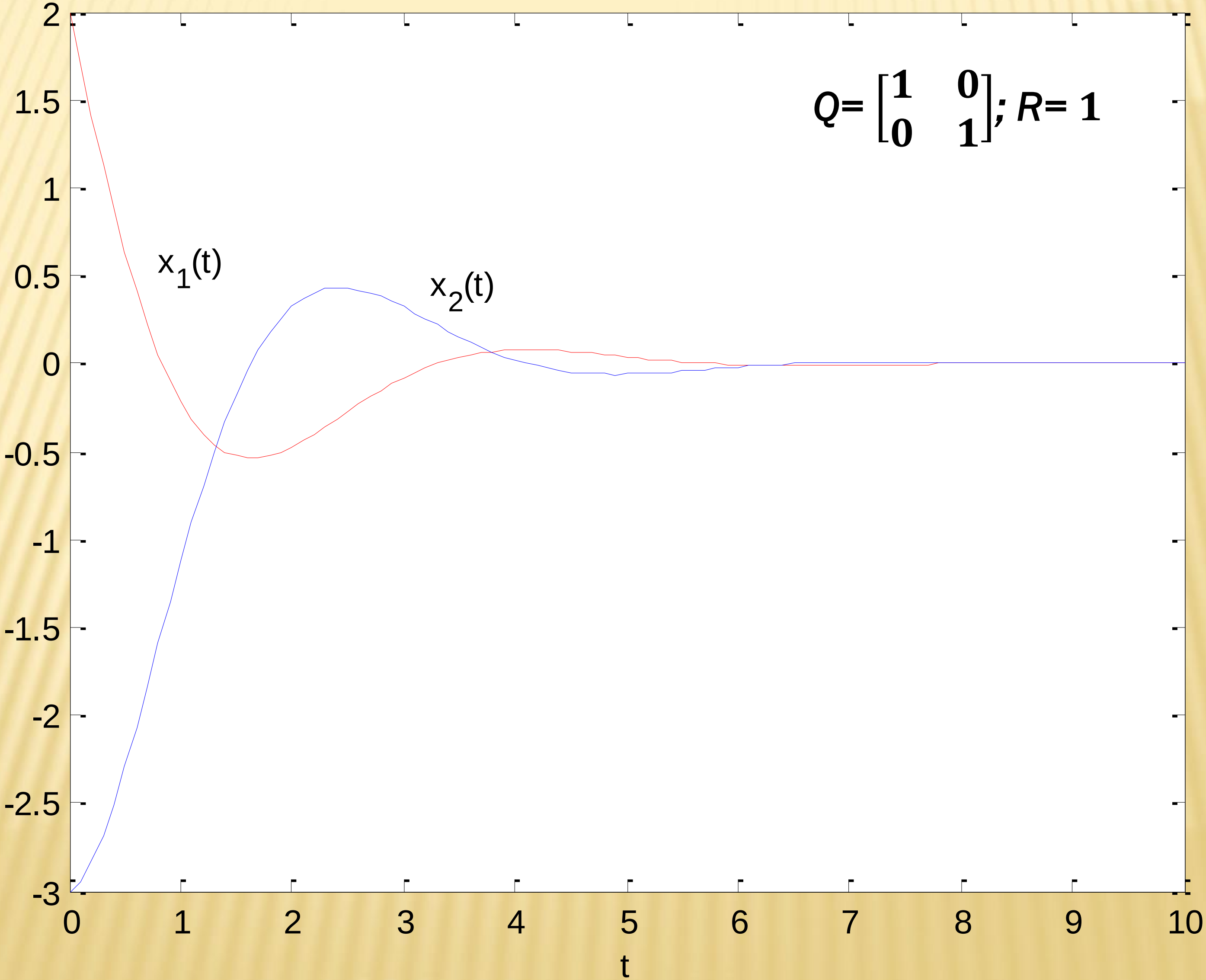
1. A simple choice is to use diagonal weights:

$$Q = \begin{bmatrix} q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n \end{bmatrix}; R = \begin{bmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_l \end{bmatrix}$$

2. The diagonal elements describe how much each state and input (squared) should contribute to the overall cost.

- ***For Example* 16, *Let*** $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; R = 1$

LINEAR QUADRATIC REGULATOR



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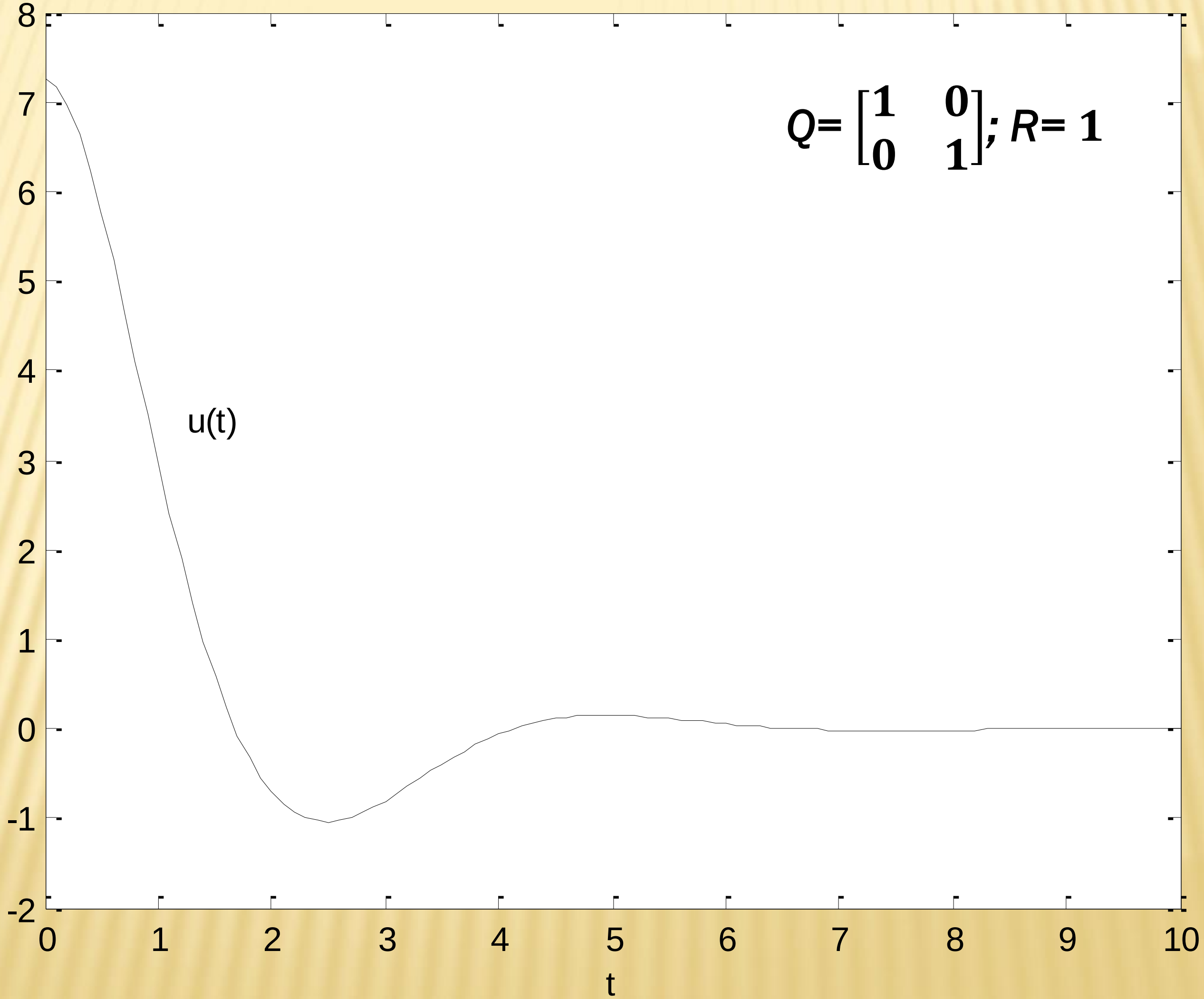

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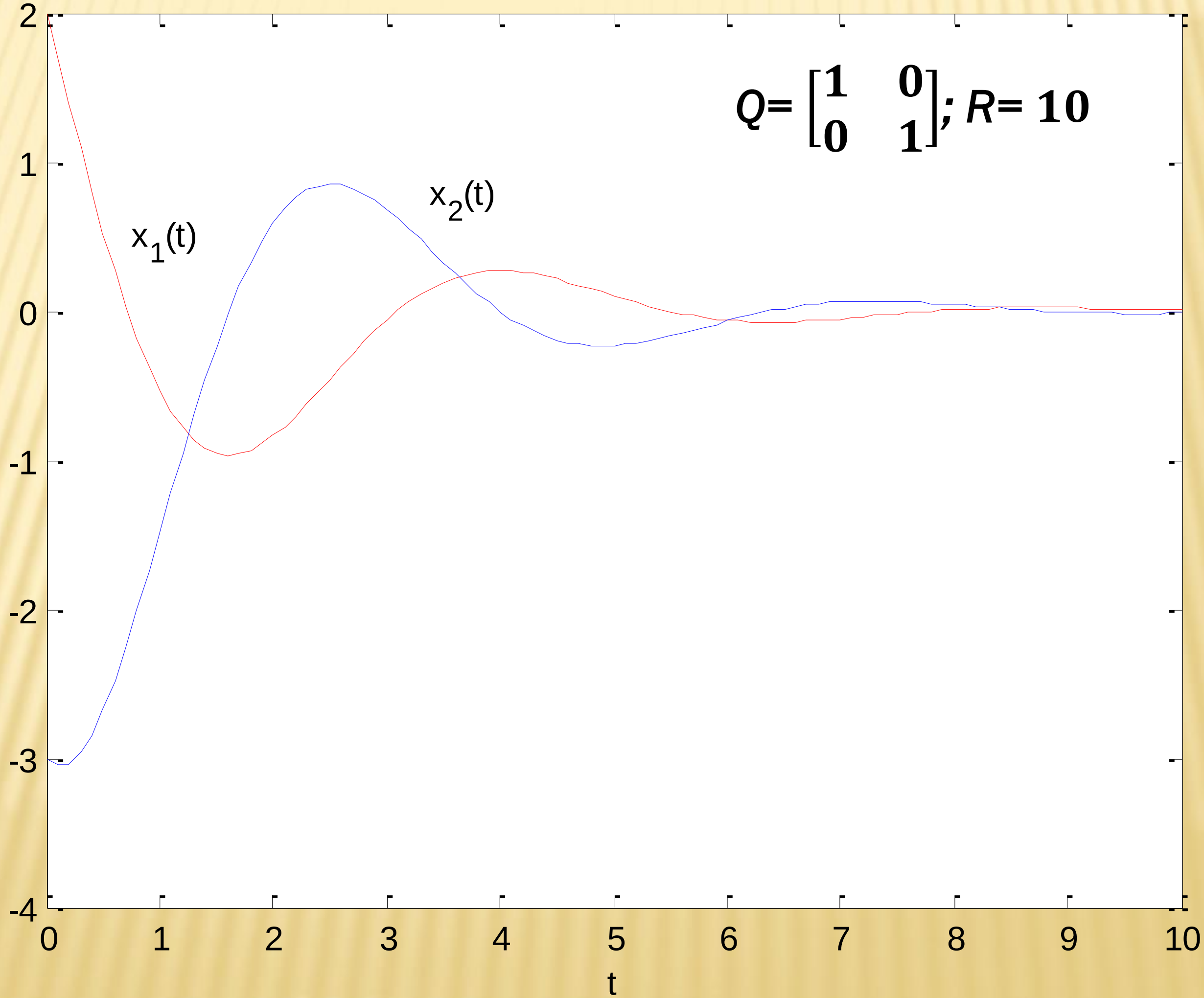
- ***For Example* 16, *Let*** $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; R = 10$



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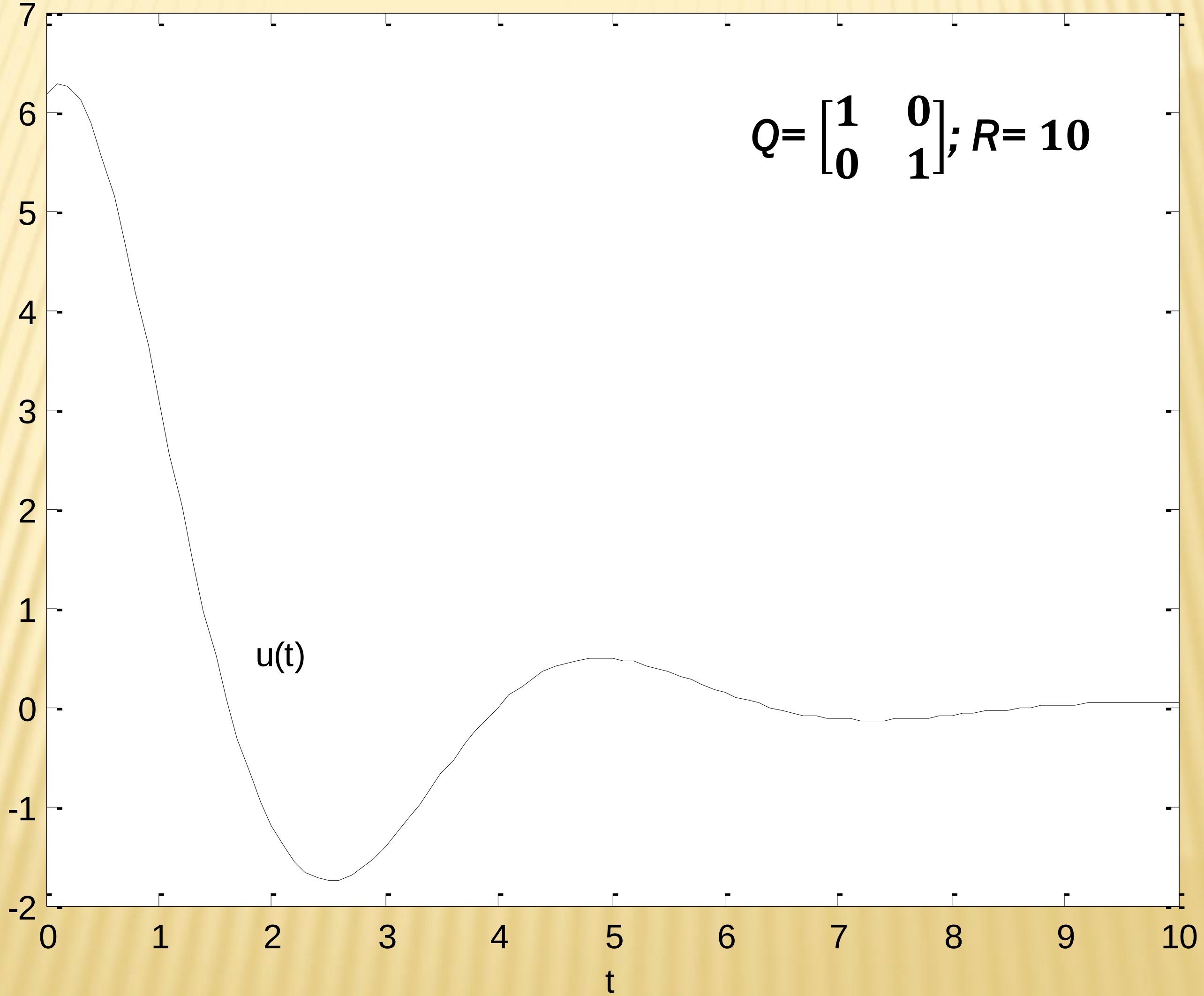

Thank You for Your Attention